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LETTER TO THE EDITOR

Inequalities on the time evolution of probabilities

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Abstract. We consider probabilities $p_M(t)$ for quantum mechanical systems to be found in a subspace M of the Hilbert space of states and derive an upper bound on the change in time of $p_M(t)$ which contains only the Hamiltonian and the projector P_M onto M but not the density matrix.

In the discussion of the solar neutrino puzzle, neutrino propagation in matter and magnetic fields plays an important role (for recent reviews see [1]). In the ultrarelativistic case, which is most likely relevant for all neutrinos participating in solutions to this problem, neutrino propagation can be studied by considering an effective Schrödinger equation [2]

$$\mathrm{i}\frac{\mathrm{d}\psi}{\mathrm{d}t} = H\psi \tag{1}$$

where ψ is a vector whose components are labelled by neutrino helicity and flavour and H is simply a t-dependent matrix acting on ψ. In the most general situation envisaged in the literature H is given by [2, 1]

$$H = \begin{pmatrix} V_{\mathrm{L}} + (1/2p)V_{\mathrm{S}}^{\dagger}V_{\mathrm{S}} & -B_{+}\lambda^{\dagger} \\ -B_{-}\lambda & V_{\mathrm{R}} + (1/2p)V_{\mathrm{S}}V_{\mathrm{S}}^{\dagger} \end{pmatrix} \tag{2}$$

for neutrinos propagating in the sun. In this matrix p is the neutrino momentum and $B_{\pm} = B_1 \pm \mathrm{i}B_2$ with B_1, B_2 being components of the magnetic field orthogonal to the propagation direction. For n_{F} flavours V_{L}, V_{R} and λ are $n_{\mathrm{F}} \times n_{\mathrm{F}}$ matrices. In the simplest case V_{L} describes coherent forward scattering by matter through the standard model charged and neutral current interactions, $V_{\mathrm{R}} = 0$ for Dirac neutrinos and $-V_{\mathrm{L}}^{*}$ for Majorana neutrinos, $V_{\mathrm{S}} = M$, the neutrino mass matrix, and λ contains the magnetic moments and transition moments [3]. In notation (2) the upper n_{F} components of ψ have negative and the lower half positive helicity. V_{L} (V_{R}) and $B_{1,2}$ vary along the neutrino path. Therefore H depends on t.

Recently Moretti [4] has derived an inequality for the probability of transitions from negative to positive helicity in the case of two neutrino flavours and the Hamiltonian (2). In this letter we will show that the result of [4] is a special case of a class of inequalities involving probabilities $p_M(t)$ for finding the system in a given hyperplane M of the Hilbert space. These inequalities are also valid if the state of the system is described by a density matrix ρ_t. In this case (1) has to be replaced by

$$\mathrm{i}\dot{\rho}_t = [H(t), \rho_t]\,. \tag{3}$$

Consequently, the general result is neither confined to probabilites for spin flip nor to the special form of H in (2). With some caution it will also be applicable to systems with an infinite number of degrees of freedom but there questions of the domain of $H(t)$ will be important. To avoid these complications we will confine ourselves to the finite-dimensional case and prove the following theorem.

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Theorem. Suppose that a system described by a density matrix ρ_t and a Hamiltonian $H(t)$ has a finite number of degrees of freedom, i.e. ρ_t and $H(t)$ act on a finite-dimensional Hilbert space $\mathcal{H}$. Furthermore, let M be a hyperplane of $\mathcal{H}$ with P_M being the projector onto it. Then the probability to find the system in M is given by

$$p_M(t) = \mathrm{Tr}\,(\rho_t P_M) \tag{4}$$

and the inequality

$$|p_M(t_1) - p_M(t_0)| \leqslant \int_{t_0}^{t_1} \mathrm{d}t (\mathrm{Tr}\,(P_M H(t)(1 - P_M)H(t)P_M))^{1/2} \tag{5}$$

holds for all $t_0 \leqslant t_1$.

Proof. It is convenient to consider first pure states with density matrices

$$\rho_t = |\psi_t\rangle\langle\psi_t|\,. \tag{6}$$

At the end it will be easy to see the validity of (5) for mixed states.

In the case of (6) the probability (4) is given by

$$p_M(t) = \langle\psi_t|P_M\psi_t\rangle\,. \tag{7}$$

To prove (5) we first take the derivative of (7):

$$\frac{\mathrm{d}P_M}{\mathrm{d}t} = 2\,\mathrm{Re}\,\langle\psi|P_M\dot{\psi}\rangle = 2\,\mathrm{Im}\,\langle\psi|P_M H\psi\rangle = 2\,\mathrm{Im}\,\langle\psi|P_M H(1 - P_M)\psi\rangle\,. \tag{8}$$

Then we integrate (8), use $|\mathrm{Im}\,z| \leqslant |z|$ and employ the Cauchy–Schwarz inequality. Thus we obtain

$$\begin{aligned} |p_M(t_1) - p_M(t_0)| &\leqslant 2\int_{t_0}^{t_1} \mathrm{d}t\,|\langle\psi|P_M H(1 - P_M)\psi\rangle| \\ &\leqslant 2\int_{t_0}^{t_1} \mathrm{d}t\,\|P_M\psi\|\,\|P_M H(1 - P_M)\psi\|\,. \end{aligned} \tag{9}$$

In the next step we calculate the length of the vector $P_M H(1 - P_M)\psi$ by using an orthonormal basis $\{e_\gamma|\gamma = 1, \ldots, \dim\mathcal{H}\}$ and employ the Cauchy–Schwarz inequality once more which leads to

$$\begin{aligned} |p_M(t_1) - p_M(t_0)| &\leqslant 2\int_{t_0}^{t_1} \mathrm{d}t\,\|P_M\psi\|\left(\sum_\gamma |\langle e_\gamma|P_M H(1 - P_M)\psi\rangle|^2\right)^{1/2} \\ &\leqslant 2\int_{t_0}^{t_1} \mathrm{d}t\,\|P_M\psi\|\left(\sum_\gamma \|(1 - P_M)HP_M e_\gamma\|^2\right)^{1/2} \|(1 - P_M)\psi\| \\ &\leqslant \int_{t_0}^{t_1} \mathrm{d}t\left(\sum_\gamma \|(1 - P_M)HP_M e_\gamma\|^2\right)^{1/2}. \end{aligned} \tag{10}$$

In the last step we have taken advantage of

$$\|(1-P_M)\psi\|^2 = 1 - \|P_M\psi\|^2 \qquad \text{and} \qquad 2\,\|P_M\psi\|\,\|(1-P_M)\psi\| \leqslant 1\,. \tag{11}$$

In this way the state vector ψ drops out of the integral.

Finally, we show that (10) is also valid for general density matrices

$$\rho = \sum_\alpha \rho_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| \tag{12}$$

where $\{\psi_\alpha|\alpha = 1, \ldots, \dim\mathcal{H}\}$ is an orthonormal basis of $\mathcal{H}$ and $\sum_\alpha \rho_\alpha = 1$ with $\rho_\alpha \geqslant 0$ for all α. In this case we define probabilities $p_M^\alpha(t) = \langle\psi_{\alpha t}|P_M\psi_{\alpha t}\rangle$ such that

$$p_M = \sum_\alpha \rho_\alpha p_M^\alpha. \tag{13}$$

Using inequality (10) for each of the p_M^α we obtain

$$|p_M(t_1) - p_M(t_0)| \leqslant \sum_\alpha \rho_\alpha |p_M^\alpha(t_1) - p_M^\alpha(t_0)| \leqslant \int_{t_0}^{t_1} \mathrm{d}t \left(\sum_\gamma \|(1-P_M)HP_Me_\gamma\|^2\right)^{1/2} \tag{14}$$

which shows the validity of the theorem for general mixed states.

Remarks. As can be seen from the proof the integrand in inequality (5) can be represented in several equivalent forms by using properties of projectors and traces. These expressions are given by

$$\begin{aligned} \mathrm{Tr}\,(P_MH(1-P_M)HP_M) &= \mathrm{Tr}\,((1-P_M)HP_MH(1-P_M)) \\ &= \mathrm{Tr}\,(P_MH(1-P_M)H) = \sum_\gamma \|(1-P_M)HP_Me_\gamma\|^2 \\ &= \sum_\gamma \|P_MH(1-P_M)e_\gamma\|^2\,. \end{aligned} \tag{15}$$

In the case of $\dim\mathcal{H} = \infty$ strong enough assumptions have to be made to allow for the manipulations at each step of the proof. In the final result the third expression in (15) will not be suited for infinite dimensions because the operator $P_MH(1-P_M)H$ will in general be neither positive nor in the trace class. For positive operators the trace is well defined if infinity is included in its range [5] and thus the other expressions in (15) will make sense apart from possible difficulties with the domain of $H(t)$. If M is finite-dimensional and all manipulations with respect to the domain of $H(t)$ are well defined in (15) then we expect (5) to hold. If $\dim M = \infty$ there will still be cases where (5) is applicable and gives a finite bound. However, it is easy to find examples even with bounded Hamiltonians such that the traces in (15) are infinite. Needless to say in this case (5) is useless.

Let us come back to the spin flip in the solar neutrino problem. The subspace of positive helicity is given by the vectors which have zeros in the first n_F components. Then the probability that the neutrino has positive helicity

$$p_+ = \sum_{j=n_F+1}^{2n_F} |\psi_j|^2 = 1 - p_- \tag{16}$$

is zero at the time of production in the sun. For simplicity we have taken a pure state ψ in (16). Therefore we obtain an upper bound for detecting a neutrino with positive helicity [4]

$$p_+^d \leqslant \sqrt{\mathrm{Tr}(\lambda^\dagger \lambda)} \left\langle \int_{x_0}^{x_1} \mathrm{d}x |\boldsymbol{B}_\perp| \right\rangle . \tag{17}$$

The integral goes over the neutrino path in the sun and $\boldsymbol{B}_\perp$ is the magnetic field orthogonal to the neutrino momentum. One has to take into account that p_+ actually depends on the path and on the magnetic field configuration along it at the time the neutrino travels through the sun. Therefore, actually a probability p_+^d averaged over neutrino energies and paths and over magnetic fields varying with path and time is measured and the brackets around the integral in (5) indicate that the integral also has to be averaged accordingly.

Inequality (5) can, of course, also be applied to other probabilities than p_+, p_- in the solar neutrino problem. However, in the case of vacuum mixing it is not so easy to extract useful information from (5) because vacuum mixing is also operative outside the sun. If the distance from sun to earth is incorporated into the integral then it is much greater than one for the usual range of mixing angles and mass differences in the solar neutrino problem and the inequality is useless. Therefore in that case $p_M(t_1)$ should be considered as a probability at the edge of the sun and neutrino oscillations should be taken into account with other methods.

In the present letter we have derived a class of inequalities for the time evolution of probabilities valid in general quantum mechanical systems. In this way we have generalized an inequality derived in [4] for two neutrino flavours and the Hamiltonian (2).

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References

[1] Kennedy D C 1991 Invited Lectures (1990 Theoretical Advanced Studies Institute, University of Colorado at Boulder) *Preprint UPR 0442 T (REV)*
BabuK S 1990 *proc. Fourth Mexican School of Particles and Fields (Oaxtepec 1990)* p 104
Pal P B 1992 *Int. J. Mod. Phys.* A 7 5387

[2] Wolfenstein L 1978 *Phys. Rev.* D **17** 2369
Mikheyev S P and Smirnov A Yu 1985 *Sov. J. Nucl. Phys.* **42** 913; 1989 *Prog. Part. Nucl. Phys.* **23** 41

[3] Voloshin M B, Vysotskii M I and Okun L B 1986 *Sov. J. Nucl. Phys.* **44** 440; 1986 *Sov. Phys.–JETP* **64** 446

[4] Moretti M 1992 *Phys. Lett.* **293B** 378

[5] Thirring W 1981 *Quantum Mechanics of Atoms and Molecules (Wien 1981)* (New York: Springer) p 48